LAMINAR JET IN "NARROW" SLOT AT LARGE REYNOLDS NUMBERS

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In view of the complexity of spatial, significantly three-dimensional problems, it appears justified to switch from a three-dimensional to two-dimensional mathematical model of the flow. The reduction of the three dimensional initial-boundary value problem for the Stokes system to successive two-dimensional problems were discussed in [1].

The study of the dynamics of viscous fluid at low and moderate Reynolds numbers is mainly related to internal flows: channel flows with plane parallel walls with sudden expansion in channel cross-section, propagation of viscous jets in space heated by the same fluid, and also problems with heat convection. It is known [2] that the general steady-state, nonlinear Navier–Stokes equations have at least one laminar solution at any Reynolds number if, for each isolated component S_k of the boundary S in region Ω filled with fluid, the following condition is satisfied:

$$\int_{S_k} \mathbf{V}\mathbf{n}\,dS = 0.$$

For a finite region and low Reynolds numbers the solution to the problem is unique and stable. Hence it is of great interest to study the case of high Reynolds numbers when the flow is still laminar.

Consider an incompressible, viscous jet between two plane parallel plates with the approximation [3] when the distance between them (2h) is much smaller than the basic dimensions (L and δ) of the problem; for the "narrow" jet h could be comparable to δ.

Approximate solution to the Navier–Stokes equation is written in the form

$$u = u_m(x,\ y,\ t)f(z/h),\ v = v_m(x,\ y,\ t)f(z/h),$$
$$w = w_m(x,\ y,\ t)\varphi(z/h),\ p = p_m(x,\ y,\ z,\ t),$$

in the steady-state case $\partial u/\partial t = 0$; then

$$u = u_m(x,\ y)f(z/h),\ v = v_m(x,\ y)f(z/h),$$
$$w = w_m(x,\ y)\varphi(z/h),\ p = p_m(x,\ y,\ z), \tag{1}$$

where f and φ are functions of the distance between plates. The solution to Navier–Stokes equations is expressed in the form of parabolic distribution of Poiseuille flow. In turbulent flow the velocity profile could have the exponential relation

$$u = u_m(z/h)^{1/k},\ v = v_m(z/h)^{1/k},\ w = w_m\varphi(z/h),\ p = p_m(x,\ y,\ z).$$

The integration of the Navier–Stokes equations is carried out along the vertical coordinate z with constant intervals from –h to h according to the relations

$$\langle u\rangle = \frac{1}{2h}\int_{-h}^{h} u\,dz,\quad \langle v\rangle = \frac{1}{2h}\int_{-h}^{h} v\,dz,\quad \langle w\rangle = \frac{1}{2h}\int_{-h}^{h} w\,dz.$$

Cartesian coordinates are used with the origin at the center between the two plates, x and y axes are in the plane parallel to the plates, and the axis z is perpendicular to the latter (Fig. 1).

We have the system

$$\frac{\partial}{\partial x}[2h\langle u^2\rangle] + \frac{\partial}{\partial y}[2h\langle uv\rangle] + [uw]_{z=-h}^{z=h} = -\frac{1}{\rho}\frac{\partial}{\partial x}[2h\langle p\rangle] + \nu\left(\frac{\partial^2}{\partial x^2}[2h\langle u\rangle] + \frac{\partial^2}{\partial y^2}[2h\langle u\rangle] + \left|\frac{\partial u}{\partial z}\right|_{z=-h}^{z=h}\right),$$

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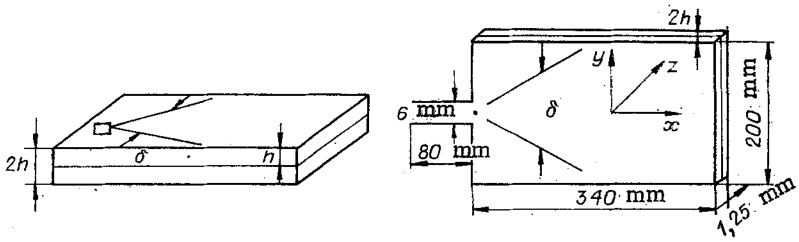

Fig. 1

$$\frac{\partial}{\partial x}[2h\langle uv\rangle]+\frac{\partial}{\partial y}[2h\langle v^2\rangle]+[vw]_{z=-h}^{z=h}=-\frac{1}{\rho}\frac{\partial}{\partial y}[2h\langle p\rangle]+\nu\left(\frac{\partial^2}{\partial x^2}[2h\langle v\rangle]+\frac{\partial^2}{\partial y^2}[2h\langle v\rangle]+\left[\frac{\partial v}{\partial z}\right]_{z=-h}^{z=h}\right),$$

$$\frac{\partial}{\partial x}[2h\langle wu\rangle]+\frac{\partial}{\partial y}[2h\langle wv\rangle]+[w^2]_{z=-h}^{z=h}=-\frac{1}{\rho}[p]_{z=-h}^{z=h}+\nu\left(\frac{\partial^2}{\partial x^2}[2h\langle w\rangle]+\frac{\partial^2}{\partial y^2}[2h\langle w\rangle]+\left[\frac{\partial w}{\partial z}\right]_{z=-h}^{z=h}\right), \quad (2)$$

$$\frac{\partial}{\partial x}[2h\langle u\rangle]+\frac{\partial}{\partial y}[2h\langle v\rangle]+[w]_{z=-h}^{z=h}=0.$$

After the substitution of profiles (1) in integrals for the steady-state flow we get the following system of equations of motion and continuity:

$$u_m\frac{\partial u_m}{\partial x}+v_m\frac{\partial u_m}{\partial y}=-\frac{15}{8\rho}\frac{\partial p}{\partial x}+\frac{5}{4}\nu\left(\frac{\partial^2 u_m}{\partial x^2}+\frac{\partial^2 u_m}{\partial y^2}\right)-\frac{15\nu}{4h^2}u_m,$$

$$u_m\frac{\partial v_m}{\partial x}+v_m\frac{\partial v_m}{\partial y}=-\frac{15}{8\rho}\frac{\partial p}{\partial y}+\frac{5}{4}\nu\left(\frac{\partial^2 v_m}{\partial x^2}+\frac{\partial^2 v_m}{\partial y^2}\right)-\frac{15\nu}{4h^2}v_m, \quad (3)$$

$$u_m\frac{\partial w_m}{\partial x}+v_m\frac{\partial w_m}{\partial y}+w_m^2[\varphi(z/h)]_{z=-h}^{z=h}=-\frac{15}{8}\frac{1}{\rho}[p]_{z=-h}^{z=h}+\frac{5}{4}\nu\left(\frac{\partial^2 w_m}{\partial x^2}+\frac{\partial^2 w_m}{\partial y^2}\right)+w_m\left[\frac{\partial}{\partial z}\varphi(z/h)\right]_{z=-h}^{z=h},$$

$$\frac{\partial u_m}{\partial x}+\frac{\partial v_m}{\partial y}+w_m[\varphi(z/h)]_{z=-h}^{z=h}=0.$$

Assuming $\partial w_m/\partial x = 0$, $\partial w_m/\partial y = 0$ and the condition for the function $w_m(x, y) = \text{const}$, it is possible to put

$$w_m=0,\ w=w_m\varphi(z/h)=0,\ p=p_m(x,\ y). \quad (4)$$

The system of equations (3) with condition (4) takes the form

$$u_m^0\frac{\partial u_m^0}{\partial x^0}+v_m^0\frac{\partial u_m^0}{\partial y^0}\frac{L}{\delta}=-\frac{15}{8}E\frac{\partial p^0}{\partial x^0}+\frac{5}{4}\frac{1}{\mathrm{Re}_L}\left(\frac{\partial^2 u_m^0}{\partial x^{02}}+\frac{\partial^2 u_m^0}{\partial y^{02}}\frac{L}{\delta^2}\right)^2-\frac{15L^2}{4h^2}\frac{u_m^0}{\mathrm{Re}_L},$$

$$u_m^0\frac{\partial v_m^0}{\partial x^0}+v_m^0\frac{\partial v_m^0}{\partial y^0}\frac{L}{\delta}=-\frac{15}{8}E\frac{\partial p^0}{\partial y^0}\frac{L}{\delta}+\frac{5}{4\mathrm{Re}_L}\left(\frac{\partial^2 v_m^0}{\partial x^{02}}+\frac{\partial^2 v_m^0}{\partial y^{02}}\frac{L^2}{\delta^2}\right)-\frac{15L^2}{4h^2}\frac{v_m^0}{\mathrm{Re}_L}, \quad (5)$$

$$\frac{\partial u_m^0}{\partial x}+\frac{L}{\delta}\frac{\partial v_m^0}{\partial y^0}=0,$$

$$u_m=u_m^0V,\quad v_m=v_m^0V,\quad x=x^0L,\quad y=y^0\delta,\quad p=p^1p^0=E\rho V^2p^0.$$

Here, $E = p^1/\rho V^2$ is the Euler number; $\mathrm{Re}_L = LV/\nu$ is the Reynolds number; $(L^2/h^2,\ L^2/\delta^2,\ L/\delta)$ are internal geometrical simplexes. If the jet is "narrow," $\delta \ll L$ and δ is of the same order as h, then it is possible to assume that the following system of equations is correct within the boundary-layer approximations:

$$u_m\frac{\partial u_m}{\partial x}+v_m\frac{\partial u_m}{\partial y}=-\frac{15}{8\rho}\frac{\partial p}{\partial x}+\frac{5}{4}\nu\frac{\partial^2 u_m}{\partial y^2}-\frac{15\nu}{4h^2}u_m,$$

$$\frac{\partial p}{\partial y}=-\frac{2\mu}{h^2}v_m,\quad \frac{\partial u_m}{\partial x}+\frac{\partial v_m}{\partial y}=0. \quad (6)$$

In the region of intermediate asymptotes, when $h < \delta \ll L$, the boundary-layer phenomenon as a region of large gradients in functions occurs within narrow zones near the parts of the boundary where there is a difference between the number of boundary conditions in the basic system of equations and the singular problem (when the small parameter is zero), and in our case this term is $(5\nu/4)\partial^2 u_m/\partial y^2$.

The method of integral relations along with approximate methods [4] using polynomial approximation for shear stress and approximation for pressure were used to complete jet characteristics. The shear stress profile in the jet is given in the form of a polynomial

$$\tau = \sum_{i=1}^{3} A_i y^{(i-1)} = A_1 + A_2 y + A_3 y^2,$$

whose coefficients are determined from boundary conditions at the jet axis y = 0 and at its boundaries. In view of symmetry we have

$$\tau = 0 \text{ for } y = 0,\ A_1 = 0;$$
$$\tau = 0 \text{ for } y = \delta,\ A_2 = -A_3\delta,$$

then

$$\tau = A_2 y + A_3 y^2 = A_2(y - y^2/\delta) = A_2\delta(\eta - \eta^2),\ \eta = y/\delta. \tag{7}$$

The expression (7) for the shear stresses profile makes it possible to close the problem and determine the velocity profile in the jet:

$$\frac{\partial u_m}{\partial y} = \frac{A_2\delta}{\mu}(\eta - \eta^2). \tag{8}$$

After integrating (8), when the constant of integration C is found from conditions on the axis $u_m = u_{mm}$ at y = 0, we have

$$(u_m - u_{mm}) = \frac{A_2}{6\mu}\delta^2(3\eta^2 - 2\eta^3), \quad \eta = \frac{y}{\delta}. \tag{9}$$

At the jet boundary y = δ, η = 1, and the velocity u_m = 0. From (9) we find that when η = 1 the polynomial coefficient

$$A_2 = -\frac{6\mu}{\delta^2} u_{mm}. \tag{10}$$

Simultaneous solution of Eqs. (9) and (10) gives the velocity profile

$$F(\eta) = \frac{u_m}{u_{mm}} (1 - 3\eta^2 + 2\eta^3). \tag{11}$$

The coefficient A_2 may be determined from the equations of motion and Eq. (10) in the form

$$\left[\frac{\partial\tau}{\partial y}\right]_{y=0} = A_2 = -\frac{6\mu}{\delta^2} u_{mm} = \frac{4}{5}\rho u_{mm}\frac{du_{mm}}{dx} + \frac{3dp}{2dx}\bigg|_{y=0} + \frac{3\mu u_{mm}}{h^2},$$

which leads to an expression for the pressure gradient

$$\frac{dp}{dx}\bigg|_{y=0} = \frac{8\rho}{15} u_{mm}\frac{du_{mm}}{dx} + 2\mu u_{mm}\left(\frac{2}{\delta^2} + \frac{1}{h^2}\right). \tag{12}$$

Pressure p is approximated by the following conditions on the jet axis: p = p_m at y = 0, ∂p/∂y = 0 at y = 0, since v_{mm} = 0. At the jet boundary p = p_1 when y = δ, where p_1 = const. Consider the polynomial

$$p = \sum_{i=1}^{3} B_i y^{(i-1)} = B_1 + B_2 y^2 + B_3 y^3,$$

whose coefficients are determined according to the given conditions. Then

$$p = p_m + (p_1 - p_m)\eta^2,\ \eta = y/\delta \tag{13}$$

with

$$B_1 = p_m,\ B_3 = (p_1 - p_m)/\delta^2,\ B_2 = 0.$$

Using the equation for nondimensional velocity profile (11) and Eq. (13) for pressure we integrate the first of the basic equations of motion (6) across the jet from the axis of the free jet to the outer boundary, from y = 0 to y = δ, taking into consideration that the transverse velocity v_m = 0 on the jet axis. The first integral relation is obtained in the form

$$\frac{13}{35}\frac{\partial}{\partial x}\left[u_{mm}^2\delta\right] = -\frac{15}{8}\frac{\nu}{h^2}[u_{mm}\delta] + \frac{5}{4\rho}\frac{\partial}{\partial x}[\delta(p_1 - p_m)]. \tag{14}$$

Multiplying the terms of the first of Eqs. (6) by u_m and integrating across the jet along y, we have the second integral relation

$$\frac{43}{280}\frac{\partial}{\partial x}\left[u^3_{mm}\delta\right] = -\frac{39}{28}\frac{\nu}{h^2}\left[u^2_{mm}\delta\right] - \frac{\nu u^2_{mm}}{4\delta} + \frac{13}{16\rho}\delta u_{mm}\frac{\partial (p_1 - p_m)}{\partial x} + \frac{u_{mm}d\delta}{4\rho\, dx}(p_1 - p_m). \tag{15}$$

The first integral relation for the second equation in the given system (6) is written in the form

$$(p_1 - p_m) = \frac{\mu}{h^2}\delta^2\frac{\partial u_{mm}}{dx}\frac{7}{10} + \frac{\mu}{h^2}\delta u_{mm}\frac{d\delta}{dx}\frac{2}{5}. \tag{16}$$

The system of Eqs. (12), (14)-(16) is solved simultaneously. After the substitution of relations for the pressure difference $(p_1 - p_m)$ and the derivative of pressure difference $d(p_1 - p_m)/dx$ in the first and second-order integral relations (15) and (14) obtained from the first equation of motion and after carrying out a few simple transformations, we get a system of two ordinary first-order differential equations

$$\begin{gathered} F(\delta) = \frac{d\delta}{dx} = \frac{A}{2} + \sqrt{\left(\frac{A}{2}\right)^2 - B}, \\ \frac{du_{mm}}{dx} = -u_{mm}\frac{d\delta}{dx}\frac{1}{\delta}\frac{111}{17} + \frac{\nu 2800}{\delta^2 17} + \frac{150\nu}{17h^2}, \end{gathered} \tag{17}$$

where $A = \dfrac{7\cdot 49\cdot 400\nu^2h^2 + u^2_{mm}h^4 120 + 49\cdot 150\nu^2\delta^2}{7\cdot 709\nu h^2 u_{mm}\delta}$; $B = \dfrac{3080h^2 7 + 135\delta^2}{21\cdot 709\delta^2}$. This system leads to the solution of nonlinear differential equations with given initial conditions, i.e., to a Cauchy problem. Taking the relation $\dfrac{du_{mm}}{dx} = \dfrac{du_{mm}}{d\delta}\dfrac{d\delta}{dx}$ into account, the system (17) is expressed in the form

$$\begin{gathered} \frac{du_{mm}}{d\delta} = -\frac{u_{mm}}{\delta}\frac{111}{17} + \frac{\nu 2800}{\delta^2 17F(\delta)} + \frac{150\nu}{17h^2F(\delta)}, \\ F(\delta) = \frac{A}{2} + \sqrt{\left(\frac{A}{2}\right)^2 - B}. \end{gathered} \tag{18}$$

There are stable numerical methods with high-order approximations [3] to solve the Cauchy problem.

The equation for pressure differences $(p_1 - p_m)$ is transformed to the following form using the second equation of the system (17)

$$(p_1 - p_m) = -\frac{709\delta}{170h^2}\mu u_{mm}\frac{d\delta}{dx} + \frac{\mu\nu}{h^4}\delta^2\frac{105}{17} + \frac{\mu\nu}{h^2}\frac{35\cdot 56}{17}.$$

With an accuracy up to the constant p_1 = const the pressure at the jet axis could be computed using this equation along with Eqs. (17) and (18). It is possible to compute the gradient of pressure differences (12) on the jet axis in a similar manner.

Transverse similarity profiles are shown in Figs. 2-4. Curves 1; 1, 2; and 1-3 correspond, respectively, to $Re = u_{mm}2h/\nu$ = 130; 300; and 3300.

Figure 5 shows results computed with four-step, fourth-order, explicit Runge–Kutta scheme for streamwise velocity profiles on the axis of the free jet that expands downstream (curves 1, 3, and 5).

V. D. Zhak carried out experiments at the laboratory of the Institute of Technical Physics, Siberian Branch of the Academy of Sciences of the USSR, on a model with characteristic dimensions [L_x = 340 mm, L_y = 200 mm, 2h = 1.25 mm, slot width a = 6 mm, nozzle length L_s = 80 mm (see Fig. 1)] to measure velocity profiles in a hot jet. Similarity in the measurements of transverse velocity profiles (see Figs. 2-4) along the coordinate axes u_{mm}/u^0_{mm} (u^0_{mm} is the nozzle exit velocity) and along the abscissa $y' = y/a$ is well established, and in particular, at nondimensional $x' = x/L$, respectively, for Re = 130; 300; and 3300, x' = 1.7 (point I), x' = 6.7 (point II), and x' = 1.5 (point III).

The theoretical transverse similarity profile is determined in coordinates $F(\eta)$, $\eta = y/\delta$. For streamwise similarity profiles near the origin of the coordinate system we observe an asymptotic increase in velocity profile ($u' \to \infty$ as $x' \to 0$).

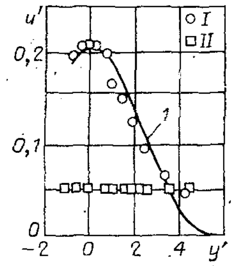

Fig. 2

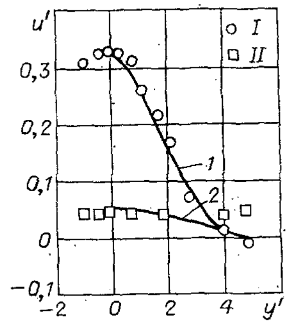

Fig. 3

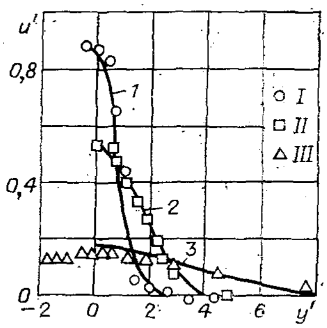

Fig. 4

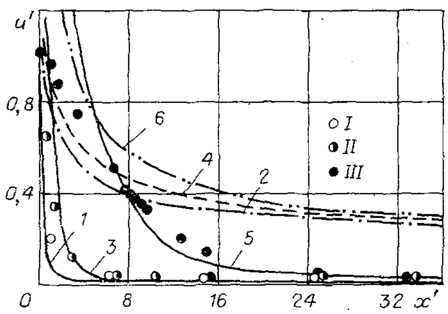

Fig. 5

At certain distance from the origin, increasing with Reynolds number, the similarity is confirmed by experimental data (Fig. 5, where curves 1, 2, and point I correspond to Re = 130; 3, 4 and point II correspond to Re = 300; 5, 6 and point III correspond to Re = 3300).

A streamwise velocity profile for plane hot jet issuing from a narrow slot into an unbounded region [1] is also shown in Fig. 5 (curves 2, 4, and 6).

For a laminar, plane jet, unbounded along the axis z, Schlichting and Bickley [3] obtained the following velocity distribution:

$$u = 0.4543(K^2/\nu x)^{1/3}(1 - \mathrm{th}^2\xi),$$
$$v = 0.5503(K\nu/x^2)^{1/3}(2\xi(1 - \mathrm{th}^2\xi) - \mathrm{th}\xi),$$
$$\xi = 0.2752(K/\nu^2)^{1/3}y/x^{2/3},$$

where K = J/ρ is the kinematic jet momentum; J is the momentum flux, which is prescribed constant for the given jet and proportional to excess pressure under which the jet issues out of the slot.

The approximate computation based on Eqs. (12), (16), and integral relations (14), (15), gives a sufficiently good agreement with the experiment. The plane, two-dimensional unbounded jet issues more rapidly from the origin, according to the streamwise profile (see Fig. 5), than the jet compressed between plates.

There are actually two zones: the rapid and the slow issue of "thin" fluid jet.

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